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# M.Sc. 1st Semester Examination-2022-23

## MATHEMATICS

**Course ID : 12153** **Course Code : MATH/103C**

**Course Title : Real Analysis**

*Time* : 2 Hours *Full Marks* : 40

*The figures in the right hand margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

*Notations and symbols have their usual meaning.*

Answer **any five** questions : 8×5=40

1. (a) Prove that $m^*$ is invariant under translation. 2

(b) If $P^*$ is a refinement of partition $P$ of $[a, b]$, prove that $L[P, f, \alpha] \le L[P^*, f, \alpha] \le U[P^*, f, \alpha] \le U[P, f, \alpha]$. 2

(c) Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases} x^2\cos\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ is of bounded variation. 2

*(Turn Over)*

(d) Show that a countable set is measurable with measure zero. 2

2. (a) Let $f$ be a function defined on a measurable set E. Prove that $f$ is measurable if and only if for any open set $G$ in $\mathbb{R}$, $f^{-1}(G)$ is measurable. 3

(b) If $f:[a, b] \to \mathbb{R}$ is monotonic and $\alpha : [a, b]$ is continuous and nondecresing, prove that $f \in \mathbb{R}(\alpha)[a, b]$. 3

(c) Give an example with justification of a decreasing sequence $\{E_n\}$ of measurable sets such that $m(\cap_n E_n) \neq \lim_{n \to \infty} m(E_n)$. 2

3. (a) Let $E \subset [0, 1)$ be measurable, $y \in [0, 1)$ and $E \oplus y = \{x + y \mid x \in E\}$. Prove that $E \oplus y$ is measurable and $m(E \oplus y) = m(E)$. 4

(b) Let $n = k$ [illegible], $0 \leq k < 2^m$ and $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = 1$ i[illegible] $\in \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right]$ and $= 0$ otherwise. Show that $f_n \to 0$ in measure for all n but the sequence $\{f_n(x)\}$ does not converge in measure for any $x$. 3

(c) Is the box function measurable ? Justify your answer. 1

4. (a) Let $E$ be any set with $m^*(E) < \infty$. Prove that $E$ is measurable if and only if there exists a measurable set $B \supset E$ such that $m(B) = m^*(E)$. 3





(b) If the function $f:[a, b] \to \mathbb{R}$ is continuous and $\alpha$ is of bounded variation on $[a, b]$, prove that the integral $\int_a^b f d\alpha$ exists. 3

(c) If the functions $f, g : [a, b] \to \mathbb{R}$ are BV on $[a, b]$, prove that $V_{fg} \leq AV_f + BV_g$ where $A = \sup\{|g(x)|\}$, $B = \sup\{|f(x)|\}$. 2

5. (a) Let $S$ be a non-measurable set with $m^*(S) < \infty$ and $G$ be $G_\delta$-set with $S \supset G$ such that $m^*(S) = m(G)$. Prove that $G - S$ is non-measurable and $m(G) < m^*(S) + m^*(T)$ where $T = G - S$. 3

(b) Evaluate $\int_{-1}^{1} (x^2 + 2) dg$ where $g(x) = 1$ for $x \geq 0$ and $= -1$ for $x < 0$. 3

(c) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{1}{1-x}, & x \neq 1 \\ 0, & x = 1 \end{cases}$ is of BV on any closed subinterval of $(0, 1)$ but $f$ is not BV on $(0, 1)$. 2

6. (a) Let $E$ be a set with $m(E) < \infty$ and $\{f_n\}$ be a sequence of measurable functions on $E$ which converges to a real-valued function $f$ a.e. on $E$. Prove that for given $\epsilon > 0$ and $\delta > 0$, there exists a measurable set $A \subset E$ with $m(A) < \delta$ and $n_0 \in \mathbb{N}$ such that for all $x \in E - A$ and $|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$. 4

(b) If $f \in \mathbb{R}(\alpha)[a, b]$ and $a<c<b$, prove that $f \in \mathbb{R}(\alpha)[a, c]$, $f \in \mathbb{R}(\alpha)[c, b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$ 4

7. (a) Let $E$ be a measurable set and the function $f : E \to \mathbb{R}$ be measurable. Prove that $|f|$ is also mesurable. Is the converse true? Answer with justification. 2+2

(b) Let $A$ be any subset of $\mathbb{R}$ and $E_1, E_2, \ldots\ldots, E_n$ be a finite collection of pairwise disjoint measurable sets. Prove that $m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$ 4

8. (a) Let $E$ be a measurable set. Prove that for a given $\epsilon > 0$, there exists a closed set $F \subset E$ such that $m^*(E - F) < \epsilon$.

3

(b) Using measure theory, prove that the closed interval $[a, b]$ is uncountable. 2

(c) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $\sum_{i=1}^{n} f(t_i)\Delta\alpha_i$ tends to finite limit $l$ as the norm $||P||$ of the subdivision $P = \{a = x_0, x_1, \ldots, x_n = b\}$ tends to 0 for any $t_k \in [x_{k-1}, x_k]$, prove that $f \in \mathbb{R}(\alpha)[a, b]$ and $\int_a^b f d\alpha = l$. 3

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